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On a new class of solutions of Painlevé equations

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The so-called exact WKB analysis, that is the analysis based on the systematic use of Borel resummed WKB solutions, provides us with a new tool in the global study of differential equations containing a large parameter. An evidence is our recipé for computing the monodromy groups of second-order Fuchsian equations; making use of exact WKB analysis, we find that the monodromy group can be expressed in terms of characteristic exponents at regular singular points and some contour integrals (related to WKB solutions) on the Riemann surface determined by the operator in question. Furthermore exact WKB analysis is applicable also to the problem of monodromy preserving deformations; the condition that the monodromy of a Fuchsian equation should be preserved inevitably causes a degeneracy of the Riemann surface and of the contour integrals on it. Since the condition for monodromy preserving deformations is described by the associated Painlevé equation, this phenomenon suggests the possibility of exact WKB analysis for Painlevé equations, which is the main subject of our recent research.

In our formulation a large parameter η is introduced also into Painlevé equations, the explicit form of which is as follows:

$$P_I: \quad \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t).$$

$$P_{II}: \quad \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + \alpha).$$

$$P_{III}: \quad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + 8\eta^2\left[2\alpha_\infty\lambda^3 + \frac{\alpha'_\infty}{t}\lambda^2 - \frac{\alpha'_0}{t} - 2\frac{\alpha_0}{\lambda}\right].$$

$$P_{IV}: \quad \frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{2}{\lambda} + 2\eta^2\left[\frac{3}{4}\lambda^3 + 2t\lambda^2 + \left(t^2 + 4\alpha_1\right)\lambda - \frac{4\alpha_0}{\lambda}\right].$$

$$\begin{aligned} P_V: \quad \frac{d^2\lambda}{dt^2} &= \left(\frac{1}{2\lambda} + \frac{1}{\lambda - 1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(2\lambda - \frac{1}{2\lambda}\right) \\ &\quad + \eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\left[(\alpha_0 + \alpha_\infty) - \frac{\alpha_0}{\lambda^2} + \frac{\alpha_2 t}{(\lambda-1)^2} - \frac{\alpha_1 t^2(\lambda+1)}{(\lambda-1)^3}\right]. \end{aligned}$$

$$\begin{aligned} P_{VI}: \quad \frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} \\ &\quad + \frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[1 - \frac{\lambda^2 - 2t\lambda + t}{4\lambda^2(\lambda-1)^2}\right. \\ &\quad \left. + \eta^2\left\{(\alpha_0 + \alpha_1 + \alpha_t + \alpha_\infty) - \frac{\alpha_0 t}{\lambda^2} + \frac{\alpha_1(t-1)}{(\lambda-1)^2} - \frac{\alpha_t t(t-1)}{(\lambda-t)^2}\right\}\right]. \end{aligned}$$

As is easily seen, these equations P_J ($J = I, \ldots, VI$) admit formal power series solutions with respect to η^{-1} determined in an algebraic manner. The existence of such formal power series solutions is actually related to the degeneracy in monodromy preserving deformations mentioned above. Now our first assertion is that not only these solutions but also the following formal solutions with exponential perturbative terms should exist for P_J:

$$\lambda = \lambda^{(0)}(t, \eta^{-1}) + e^{-\phi(t)\eta}\lambda^{(1)}(t, \eta^{-1}) + e^{-2\phi(t)\eta}\lambda^{(2)}(t, \eta^{-1}) + \cdots,$$

where each $\lambda^{(j)}(t, \eta^{-1}) = \sum_k \eta^{-k}\lambda_k^{(j)}(t)$ is a formal power series of η^{-1}, in particular

$\lambda^{(0)}(t, \eta^{-1})$ itself is a solution of P_J. Note that in this expression $\phi(t)$ and every coefficient $\lambda_k^{(1)}(t)$ of $\lambda^{(1)}$ should satisfy some first-order differential equations and that the other $\lambda_k^{(j)}$'s ($j \geq 2$) can be determined uniquely by $\lambda^{(0)}$, $\lambda^{(1)}$ and ϕ in an algebraic manner.

Taking account of these formal solutions, we can develop exact WKB analysis for Painlevé equations in a pretty satisfactory manner. More concretely, we first introduce the notion of "turning points" and "Stokes curves" for Painlevé equations, and then consider the connection formula of $\lambda^{(0)}$. In the expression of the connection formula the above formal solutions with exponential terms play an important role. In fact, for the first Painlevé equation P_I the connection formula of $\lambda^{(0)}$ can be written down quite explicitly in terms of these formal solutions. As for the other Painlevé equations, we can also discuss them by making use of some reduction theorem; near a point on Stokes curves the Painlevé equation P_J ($J = II, \ldots, VI$) can be locally transformed to P_I (more precisely, the formal solutions of P_J are transformed to those of P_I). For the details we refer to our forthcoming article on the structure of Painlevé transcendents (cf. the résumé article in the RIMS Kôkyûroku "Microlocal Analysis — Today and Future").